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Probability of Collision Between Space Objects

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The International Space Station is being designed to perform debris avoidance maneuvers based on certain criteria developed from the probability of collision $\mathcal{P}_c$. Existing methods to determine the $\mathcal{P}_c$ are based on the definition of a collision/conjunction plane that contains all of the position uncertainty associated with the problem. In this paper we develop a direct and more natural way of obtaining probability of collision and present an alternative but equivalent definition for $\mathcal{P}_c$ that leads to the same results obtained earlier. Because debris avoidance is crucial for every orbiting asset in low Earth orbit, a study of this nature helps to establish the equivalence of different methods for risk assessment and evaluation.

Introduction

THE International Space Station (ISS) shall continuously face the threat of collision with orbiting debris. Hence, there needs to be a comprehensive methodology that can assess the risks posed by individual debris encounters and suggest maneuvers when necessary. Such a study shall not only benefit the ISS, but also any future orbital asset placed in low Earth orbits. Thus, although we refer to the ISS in the rest of our paper, the analysis presented here holds true for any other orbiting asset of size and orbit comparable with that of the upcoming ISS.

Space shuttle (SS) maneuvers are commanded to avoid potential collisions with cataloged space objects (maintained by the U.S. Space Command) whenever the estimated conjunction with an object falls within a box centered on the estimated SS position. The dimensions of this conjunction box are ±5 km in the in-track direction and ±2 km in the radial and out-of-plane directions. The dimensions of such a conjunction box are probably based on prior estimates of position error covariances. The determination was made that this simple criterion, or any other known deterministic criterion[1,2] when applied to the ISS, would result in too many maneuvers.[3] In addition, unnecessary maneuvers waste fuel and hamper the microgravity experiments onboard the ISS. Although the size of the conjunction box could be decreased to decrease the manuever rate, such a step clearly increases the risk to unacceptable levels. Therefore, the ISS needs a more rigorous probability-based approach for collision avoidance.[4]

The calculation of the probability of collision $\mathcal{P}_c$ requires the error covariance of the ephemerides of both the objects at conjunction. In Ref. 4 collision is considered as a single event, and there is just one value for the $\mathcal{P}_c$ for the given encounter. Although this approach is very elegant and useful, there exists a more direct approach by Khutorovsky et al.,[5] where the collision probability is obtained as a function of time. However, the method in Ref. 5 has an assumption that the size of the main object (station/asset) is small compared to the position uncertainty of the debris object. Although such an assumption gives reasonably meaningful insights when both the colliding objects are small, it breaks down for objects not of negligible size compared to the ISS. In this paper we show that such an assumption is not necessary and thus extend the approach taken by Khutorovsky et al.[5] to obtain the same result as given by Foster.[4]

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Derivation of Collision Probability

In this section we outline a detailed but slightly modified development of Foster's method[4] of computing $\mathcal{P}_c$. The probability of collision $\mathcal{P}_c$ for the entire encounter is defined to be the conditional probability when the minimum miss distance occurs at the estimated closest point of approach (CPA).

Let $t = 0$ at the CPA, i.e., at conjunction. Referring to Fig. 1, the nominal position vectors for the ISS and debris at estimated CPA are given by $\bar{\boldsymbol{r}}_{so}$ and $\bar{\boldsymbol{r}}_{do}$, respectively. Any set of perturbed trajectories for the ISS and debris given by $\tilde{\boldsymbol{r}}_{so}$ and $\tilde{\boldsymbol{r}}_{do}$ satisfy

$$\tilde{\boldsymbol{r}}_{so} = \bar{\boldsymbol{r}}_{so} + \boldsymbol{e}_s, \qquad \tilde{\boldsymbol{r}}_{do} = \bar{\boldsymbol{r}}_{do} + \boldsymbol{e}_d \tag{1}$$

where $\boldsymbol{e}_s$ and $\boldsymbol{e}_d$ are the uncertain variations in the position vectors for the space station and debris. Obviously, for this set of perturbed trajectories, the minimum miss-distance (conjunction) does not, in general, occur at time $t = 0$. Also seen in Fig. 1 are the nominal velocity vectors for the ISS and debris denoted by $\boldsymbol{v}_s$ and $\boldsymbol{v}_d$, respectively. Given the extremely short duration of the encounter events, certain approximations can be justified. All further developments assume the following:

1) The ISS and debris object nominal trajectories can be represented by straight lines with constant velocities during the encounter. This is justified because the time duration under consideration is no more than a few seconds.

2) There is no velocity uncertainty during the encounter. This is valid because typical velocity errors are of the order of few meters/second, and the time duration for the encounter is very small.

3) The position uncertainty during the encounter is constant and equal to the value at the estimated conjunction. This is a direct consequence of assumption 2.

4) The uncertainties in the positions of the ISS and debris are represented by Gaussian distributions. Although it is true that the assumed value of the position covariances significantly affects the computed value of the probability of collision,[6] in this paper we consider these error covariances to be true representatives of the state uncertainties at conjunction.

5) The ISS is much larger than the intercepting (debris) object so that the intercepting object can be considered a point mass.

The nominal trajectories near the estimated CPA for both objects are

$$\bar{\boldsymbol{r}}_s = \bar{\boldsymbol{r}}_{so} + \boldsymbol{v}_s t, \qquad \bar{\boldsymbol{r}}_d = \bar{\boldsymbol{r}}_{do} + \boldsymbol{v}_d t \tag{2}$$

Including the position uncertainties in the debris and station, the actual (perturbed) positions are given by

$$\tilde{\boldsymbol{r}}_s(t) = \tilde{\boldsymbol{r}}_{so} + \boldsymbol{v}_s t, \qquad \tilde{\boldsymbol{r}}_d(t) = \tilde{\boldsymbol{r}}_{do} + \boldsymbol{v}_d t \tag{3}$$

The miss-vector between the space station and the debris object is defined as

$$\begin{aligned}\tilde{\boldsymbol{\rho}}(t) &= \tilde{\boldsymbol{r}}_d(t) - \tilde{\boldsymbol{r}}_s(t) = \bar{\boldsymbol{r}}_{do} - \bar{\boldsymbol{r}}_{so} + (\boldsymbol{v}_d - \boldsymbol{v}_s)t + \boldsymbol{e}_d - \boldsymbol{e}_s \\ &= \bar{\boldsymbol{\rho}}_o + \boldsymbol{e}_d - \boldsymbol{e}_s + \boldsymbol{v}_r t = \tilde{\boldsymbol{\rho}}_o + \boldsymbol{v}_r t\end{aligned} \tag{4}$$

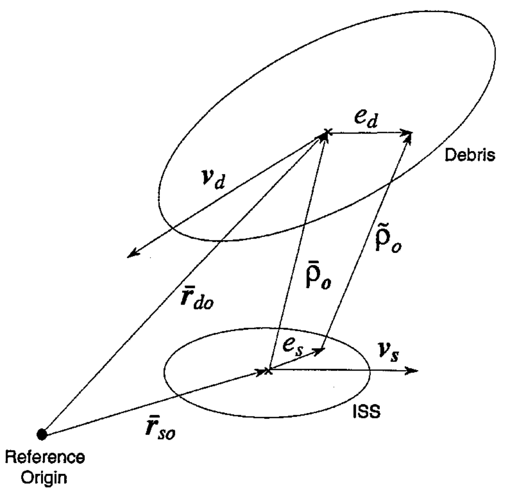

Fig. 1 Geometry of the encounter.

where $\boldsymbol{v}_r$ is the relative velocity vector and $\bar{\boldsymbol{\rho}}_o$ is the nominal miss-vector at conjunction seen in Fig. 1. These quantities are obtained by

$$\bar{\boldsymbol{\rho}}_o = \bar{\boldsymbol{r}}_{\text{do}} - \bar{\boldsymbol{r}}_{\text{so}}, \qquad \boldsymbol{v}_r = \boldsymbol{v}_d - \boldsymbol{v}_s, \qquad \tilde{\boldsymbol{\rho}}_o = \bar{\boldsymbol{\rho}}_o + \boldsymbol{e}_d - \boldsymbol{e}_s \tag{5}$$

From geometry the time of closest approach must satisfy $\mathrm{d}/\mathrm{d}t\,(\tilde{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{\rho}}) = 0$ leading to the following condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{\rho}}) = \frac{\mathrm{d}}{\mathrm{d}t}[(\tilde{\boldsymbol{\rho}}_o + \boldsymbol{v}_r t)\cdot(\tilde{\boldsymbol{\rho}}_o + \boldsymbol{v}_r t)]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\boldsymbol{\rho}}_o\cdot\tilde{\boldsymbol{\rho}}_o + 2(\tilde{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r)t + (\boldsymbol{v}_r\cdot\boldsymbol{v}_r)t^2\right]$$

$$= 2\tilde{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r + 2(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)t = 0 \tag{6}$$

Solving Eq. (6), the time of CPA is

$$t_{\text{cpa}} = -(\tilde{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r)/(\boldsymbol{v}_r\cdot\boldsymbol{v}_r) \tag{7}$$

By definition, when $\boldsymbol{e}_d = 0$, $\boldsymbol{e}_s = 0$, then $\tilde{\boldsymbol{\rho}}_o = \bar{\boldsymbol{\rho}}_o$ and $t_{\text{cpa}} = 0$. Using this in Eq. (7), we obtain

$$\bar{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r = 0 \tag{8}$$

Next, using Eq. (7), we investigate the projection of the error in the closest approach vector onto the relative velocity vector at time $t = t_{\text{cpa}}$,

$$[\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o]\cdot\boldsymbol{v}_r = (\tilde{\boldsymbol{\rho}}_o + \boldsymbol{v}_r t_{\text{cpa}} - \bar{\boldsymbol{\rho}}_o)\cdot\boldsymbol{v}_r$$

$$= (\tilde{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r) + (\boldsymbol{v}_r\cdot\boldsymbol{v}_r)t_{\text{cpa}} - \bar{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r = -\bar{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r = 0 \tag{9}$$

The preceding result is most significant in the context of this discussion because we can conclude from here that there is no uncertainty in the miss-vector in the direction of the relative velocity at the time of CPA. This result is true for all infinity of perturbed trajectories of the station and debris. The quantity t_{cpa} is a random quantity whose probability distribution can be derived from those of the space station and the debris. The result $\bar{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r = 0$ motivates Foster to define a new orthogonal coordinate system in terms of the space station and debris velocity vectors $\boldsymbol{v}_s$ and $\boldsymbol{v}_d$ and their difference $\boldsymbol{v}_r = \boldsymbol{v}_d - \boldsymbol{v}_s$ as

$$\hat{\boldsymbol{i}} = \frac{\boldsymbol{v}_r}{|\boldsymbol{v}_r|}, \qquad \hat{\boldsymbol{j}} = \frac{\boldsymbol{v}_d\times\boldsymbol{v}_s}{|\boldsymbol{v}_d\times\boldsymbol{v}_s|}, \qquad \hat{\boldsymbol{k}} = \hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}} \tag{10}$$

The $\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ plane in Fig. 2 is known as the conjunction plane.[4] Referring to Fig. 2, it is important to bear in mind that while the ISS is modeled as a sphere of radius R the figure simply depicts a projection of this sphere onto the conjunction plane. Because there is no component for the miss-vector $\bar{\boldsymbol{\rho}}_o$ in the $\hat{\boldsymbol{i}}$ direction, all of the uncertainty in the problem is restricted to the conjunction plane, and we may treat the three-dimensional problem as a two-dimensional problem.

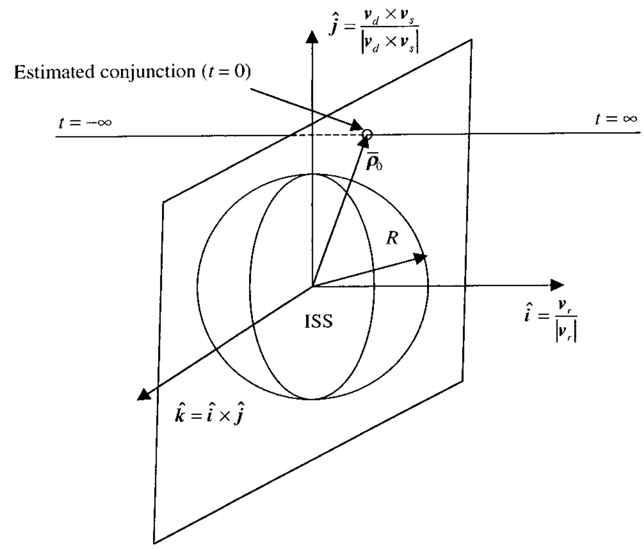

Fig. 2 Definition of the conjunction plane.

Implicit with this coordinate system, there exists an orthogonal transformation matrix C that maps this new set of unit vectors from the original coordinate system such that

$$\begin{Bmatrix}\hat{\boldsymbol{i}}\\ \hat{\boldsymbol{j}}\\ \hat{\boldsymbol{k}}\end{Bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}\begin{Bmatrix}\hat{\boldsymbol{e}}_1\\ \hat{\boldsymbol{e}}_2\\ \hat{\boldsymbol{e}}_3\end{Bmatrix}, \qquad CC^T = C^TC = I \tag{11}$$

Now, we can project the uncertainty at the CPA along these new unit vectors. The components along these directions may be obtained as

$$\alpha(t_{\text{cpa}}) = [\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o]\cdot\hat{\boldsymbol{j}}, \qquad \beta(t_{\text{cpa}}) = [\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o]\cdot\hat{\boldsymbol{k}}$$

$$\gamma(t_{\text{cpa}}) = [\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o]\cdot\hat{\boldsymbol{i}} = 0 \tag{12}$$

Notice that Eq. (9) is another way of stating that $\gamma(t_{\text{cpa}}) = 0$. Through Eqs. (4) and (5) we have already defined

$$\tilde{\boldsymbol{\rho}}(t) = \bar{\boldsymbol{\rho}}_o + \boldsymbol{e}_d - \boldsymbol{e}_s + \boldsymbol{v}_r t \tag{13}$$

So, at time $t = t_{\text{cpa}}$ we obtain

$$\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) = \bar{\boldsymbol{\rho}}_o + \boldsymbol{e}_d - \boldsymbol{e}_s + \boldsymbol{v}_r t_{\text{cpa}}$$

$$\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o = \boldsymbol{e}_d - \boldsymbol{e}_s + \boldsymbol{v}_r t_{\text{cpa}}$$

$$= \boldsymbol{e}_d - \boldsymbol{e}_s + \left[\frac{-(\tilde{\boldsymbol{\rho}}_o\cdot\boldsymbol{v}_r)}{(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)}\right]\boldsymbol{v}_r$$

$$= \boldsymbol{e}_d - \boldsymbol{e}_s - \left[\frac{(\bar{\boldsymbol{\rho}}_o + \boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\boldsymbol{v}_r}{(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)}\right]\boldsymbol{v}_r$$

$$= \boldsymbol{e}_d - \boldsymbol{e}_s - \left[\frac{(\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\boldsymbol{v}_r}{(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)}\right]\boldsymbol{v}_r \tag{14}$$

Using Eq. (14) in the first of equations (12), we obtain

$$\alpha(t_{\text{cpa}}) = [\tilde{\boldsymbol{\rho}}(t_{\text{cpa}}) - \bar{\boldsymbol{\rho}}_o]\cdot\hat{\boldsymbol{j}} = (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{j}} - \left[\frac{(\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\boldsymbol{v}_r}{(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)}\right]\boldsymbol{v}_r\cdot\hat{\boldsymbol{j}}$$

$$= (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{j}} - \left[\frac{(\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\boldsymbol{v}_r}{(\boldsymbol{v}_r\cdot\boldsymbol{v}_r)}\right]|\boldsymbol{v}_r|\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}} = (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{j}} \tag{15}$$

Similar to the preceding developments, we can obtain

$$\beta(t_{\text{cpa}}) = (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{k}} \tag{16}$$

Using Eq. (11) in Eqs. (15) and (16), it is easy to obtain

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \underbrace{\begin{bmatrix}-C_{21} & -C_{22} & -C_{23} & C_{21} & C_{22} & C_{23}\\ -C_{31} & -C_{32} & -C_{33} & C_{31} & C_{32} & C_{33}\end{bmatrix}}_{T}\begin{Bmatrix}\boldsymbol{e}_s\\ \boldsymbol{e}_d\end{Bmatrix} \tag{17}$$

One can easily observe that the matrix T can be factored as

$$T = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{T^*}\begin{bmatrix}-C & C\end{bmatrix} \tag{18}$$

If P_s and P_d are the 3×3 covariance matrices corresponding to the position uncertainty in the ISS and the debris respectively, from linear error theory the error covariance projected into the conjunction plane for α and β can be written as

$$P^* = T\begin{bmatrix}P_s & 0\\ 0 & P_d\end{bmatrix}T^T \tag{19}$$

Because the component of the uncertainty in the miss-vector in the direction of the relative velocity is zero $[\gamma(t_{\text{cpa}}) = 0]$, Foster[4] obtains the $\mathcal{P}_c$ for the encounter as

$$\mathcal{P}_c = \frac{1}{2\pi|P^*|^{\frac{1}{2}}}\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}}\exp(-S^*)\,\mathrm{d}z\,\mathrm{d}y \tag{20}$$

where

$$S^* = \left(\tilde{\rho}^* - \bar{\rho}_o^*\right)^T P^{*-1}\left(\tilde{\rho}^* - \bar{\rho}_o^*\right)/2 \tag{21}$$

$$\tilde{\rho}^* = T^*C\tilde{\rho}, \qquad \bar{\rho}_o^* = T^*C\bar{\rho}_o \tag{22}$$

The integral in Eq. (20) is over the circle of radius R in the conjunction plane. In the following section we develop the same expression for $\mathcal{P}_c$ taking a completely different approach, similar to that in Ref. 5.

Alternative Approach to $\mathcal{P}_c$

Given the position covariances for the ISS and debris in the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ coordinate system, we can obtain the covariance of the miss-vector $\tilde{\boldsymbol{\rho}}(t)$ in the conjunction frame coordinate system by the identity

$$P = \begin{bmatrix}-C & C\end{bmatrix}\begin{bmatrix}P_s & 0\\ 0 & P_d\end{bmatrix}\begin{bmatrix}-C^T\\ C^T\end{bmatrix} \tag{23}$$

For any given time t the probability distribution function that governs the relative motion between the ISS and debris object can be written as

$$p[\tilde{\rho}(t), t] = \left[1\Big/(2\pi)^{\frac{3}{2}}|P|^{\frac{1}{2}}\right]\exp(-S) \tag{24}$$

$$S = (\tilde{\rho} - \bar{\rho}_o)^T P^{-1}(\tilde{\rho} - \bar{\rho}_o)/2 \tag{25}$$

The probability of collision is defined as the probability that the debris object will intercept/pierce the sphere of radius R about the ISS during the encounter. Given a spherical geometry about the space station, for every trajectory that pierces into the volume of radius R there is a time interval following, which the CPA for that trajectory occurs within the sphere. Similarly, it is easy to visualize that every trajectory exiting the collision sphere would already have had a CPA within the region containing the ISS. This general idea holds only for spherical volumes and not necessarily for any other geometries, even convex regions. Hence considering an ensemble average of all possible events that have a CPA with the ISS within the spherical region is geometrically and mathematically equivalent to considering the ensemble average of all possible trajectories either entering the collision volume or leaving the collision volume.

In the time interval $(t, t + \mathrm{d}t)$ while leaving the sphere of radius R about the ISS, as seen in Fig. 3, the boundary of the ISS sphere can be crossed by a point distant no more than $\boldsymbol{v}_r^T\boldsymbol{n}\,\mathrm{d}t$. The conditional

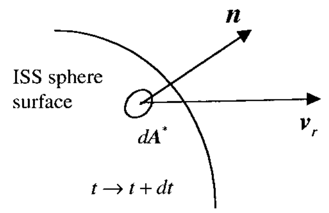

Fig. 3 Geometry of the collision during the time interval $t \to t + \mathrm{d}t$.

probability of collision in the time interval $(t, t + \mathrm{d}t)$ so that the sphere is intersected can be written as

$$\mathrm{d}P_c = \frac{1}{(2\pi)^{\frac{3}{2}}\,|P|^{\frac{1}{2}}}\int_A \exp(-S)\boldsymbol{v}_r\cdot\boldsymbol{n}\,\mathrm{d}t\,\mathrm{d}A^* \tag{26}$$

where A denotes the surface of the ISS sphere (Fig. 3) and $\boldsymbol{n}$ is the unit normal vector to any differential surface element $\mathrm{d}A^*$. Note that

$$\boldsymbol{v}_r\cdot\boldsymbol{n}\,\mathrm{d}A^* = v_r\hat{\boldsymbol{i}}\cdot\boldsymbol{n}\,\mathrm{d}A^* = v_r\,\mathrm{d}A = v_r\,\mathrm{d}y\,\mathrm{d}z \tag{27}$$

Following this, we can define the probability of collision $\mathcal{P}_c$ for the entire encounter to be

$$\mathcal{P}_c = \int_{t=-\infty}^{t=\infty}\mathrm{d}P_c = \frac{v_r}{(2\pi)^{\frac{3}{2}}|P|^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_c \exp(-S)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t \tag{28}$$

The expression for $\mathcal{P}_c$ in Eq. (28) looks different from the one in Eq. (20). In the following developments we prove their mathematical equivalence.

For notational compactness we define

$$\boldsymbol{u} = \tilde{\rho}(t) - \bar{\rho}_o \tag{29}$$

$$\boldsymbol{v} = \tilde{\rho}^* - \bar{\rho}_o^* \equiv [\alpha(t_{\text{cpa}}), \beta(t_{\text{cpa}})]^T \tag{30}$$

From Eq. (13) we can write the components of vector $\boldsymbol{u}$ to be

$$\boldsymbol{u} = [(\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{i}} + v_r t, (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{j}}, (\boldsymbol{e}_d - \boldsymbol{e}_s)\cdot\hat{\boldsymbol{k}}]^T \tag{31}$$

By virtue of the property of the conjunction plane, the quantity $\boldsymbol{v}$ is constant with time, whereas $\boldsymbol{u}$ is linearly dependent on time [from Eq. (4)]. To prove equivalence between Eq. (20) and Eq. (28), we have to show

$$\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\boldsymbol{u}^T P^{-1}\boldsymbol{u}\right)\mathrm{d}t = \frac{\sqrt{2\pi}}{v_r}\sqrt{\frac{|P|}{|P^*|}}\exp\left(-\frac{1}{2}\boldsymbol{v}^T P^{*-1}\boldsymbol{v}\right) \tag{32}$$

For any 3×3 symmetric positive definite matrix P such that

$$P = \begin{bmatrix}\eta^2 & \boldsymbol{w}^T\\ \boldsymbol{w} & P^*\end{bmatrix}, \qquad \eta \in \mathcal{R},\ \boldsymbol{w}\in\mathcal{R}^2 \tag{33}$$

we present the following identity:

$$P^{-1} = \begin{bmatrix}0 & 0\\ 0 & P^{*-1}\end{bmatrix} + \frac{|P^*|}{|P|}\begin{bmatrix}1 & -\boldsymbol{w}^T P^{*-1}\\ -P^{*-1}\boldsymbol{w} & P^{*-1}\boldsymbol{w}\boldsymbol{w}^T P^{*-1}\end{bmatrix} \tag{34}$$

This expression for P^{-1} is readily verified by using it to confirm that $PP^{-1} = I_{3\times 3}$. The elegant identity of Eq. (34) is not restricted to just 3×3 matrices, but holds for all $n\times n$ symmetric positive definite matrices; it is closely related and derivable from the matrix inversion lemma of Junkins and Kim.[7] Using the identity in Eq. (34), it is easy to factor the quadratic form

$$\boldsymbol{u}^T P^{-1}\boldsymbol{u} = \boldsymbol{v}^T P^{*-1}\boldsymbol{v} + \frac{|P^*|}{|P|}\boldsymbol{u}^T\begin{bmatrix}1 & -\boldsymbol{w}^T P^{*-1}\\ -P^{*-1}\boldsymbol{w} & P^{*-1}\boldsymbol{w}\boldsymbol{w}^T P^{*-1}\end{bmatrix}\boldsymbol{u} \tag{35}$$

Making use of Eq. (35) along with Eqs. (29) and (30) in the left-hand side of Eq. (32) and recognizing that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda^2\right) d\lambda = \sqrt{2\pi} \tag{36}$$

we indeed obtain the right-hand side of Eq. (32), thus verifying the mathematical equivalence of the two approaches to obtaining the probability of collision $\mathcal{P}_c$ from Eqs. (20) and (28).

Conclusions

Two different approaches for obtaining the probability of collision of the ISS with any debris object are discussed. The notion of computing $\mathcal{P}_c$ as an ensemble average of all possible trajectories at minimum miss-distance is geometrically very appealing while the direct approach presented in this paper establishes the collision event as a process evolving in time and space. The underlying mathematical treatment is very straightforward, and we prove that both approaches are equivalent in the sense that they yield the same formula for the probability of collision.

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